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# On Basins of Attraction for a Predator-Prey Model Via Meshless Approximation

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**Abstract.** In this work an epidemiological predator-prey model is studied. It analyzes the spread of an infectious disease with frequency-dependent and vertical transmission within the predator population. In particular we consider social predators, i.e. they cooperate in groups to hunt. The result is a three-dimensional system in which the predator population is divided into susceptible and infected individuals. Studying the dynamical system and bifurcation diagrams, a scenario was identified in which the model shows multistability but the domain of attraction of one equilibrium point can be so small that it is almost the point itself. From a biological point of view it is important to analyze this effect in order to understand under which conditions the population goes extinct or survives. Thus we present a study to analyze the basins of attraction of the stable equilibrium points. This paper addresses the question of finding the point lying on the surface which partitions the phase plane. Therefore a meshless approach has been adopted to produce an approximation of the separatrix manifold.

## INTRODUCTION

In many biological systems animals exhibit social behavior, i.e. to improve their skills in defense or to hunt they cooperate with other members of their species. Often this cooperation is a reflection of harsh environmental conditions or because of an unexpected climatic change. Many predators become pack hunters to provide more resources for the entire group. In the past decade this behavior caught the interest of ecologists and biologists because it is well-known to induce a *strong Allee effect* [1], i.e. a positive relationship between the per-capita growth rate and the population density. However, at low population densities the strong Allee effect induces extinction.For this reason many studies have been performed to understand the dynamics that drive the Allee effect, especially on endangered ecosystems with one or more species at the brisk of extinction. The aim of this work is to analyze the impact of the Allee effect induced by pack hunting, and how the system dynamics changes when an infectious disease is introduced in the predators population. In the first section we show the results obtained by analyzing the equilibrium points and bifurcation diagrams of the three-dimensional predator-prey model.Then we present an analysis of the basins of attraction to show how the Allee threshold changes, i.e. we find the critical density below which the predators go extinct. Thus, inspired by the work in Refs. [2] and [3], we develop a MATLAB algorithm to detect the points lying on the separatrix surface and then reconstruct it with a Moving Least Squares local approximation.

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**TABLE 1.** Study about Existence and stability of the equilibria

| Equilibrium | Existence | Stability |
|---|---|---|
| $E_{000}(0,0,0)$ | Always | Unstable |
| $E_i(0,0,1)$ | Always | Unstable |
| $E_n(k,0,0)$ | Always | $\beta < k(1-\theta) + \mu$ |
| $E_{ni}(k,0,i^*)$ | $\beta > \mu + (1-\theta)k \vee \beta < \mu$ | $\beta > \mu + (1-\theta)k$ |
| $E_{np}(n^*,p^*,0)$ | - | - |
| $E_{npi}(n^*,p^*,i^*)$ | $\beta > \mu + 1 - \theta \vee \beta < \mu\theta$ | - |

## ECO-EPIDEMIOLOGICAL PREDATOR-PREY MODEL

We consider the following three-dimensional eco-epidemiological model in nondimensional form:

$$\begin{aligned} \dot{n} &= r\left(1 - \frac{n}{k}\right)n - (1+\alpha p)\,np, \\ \dot{p} &= -(1+\mu i)\,p + (1+\alpha p)\,np, \\ \dot{i} &= i\,(1-i)\,(\beta-\mu) - (1+\alpha p)\,(1-\theta)\,ni. \end{aligned} \tag{1}$$

Here $n$, $p$, $i$ are respectively the prey density, the predator density and the prevalence, i.e. the proportion of the predators being infected, with $0 \leq i \leq 1$. To construct the system we start from a classical Lotka-Volterra model with logistic prey growth, where $r$ is the per-capita growth rate and $k$ the carrying capacity. Parameter $\alpha$ increases the predation rate and represents the strength of the cooperation in pack hunting. The disease is transmitted both horizontally, with transmission parameter $\beta$, and vertically, where $\theta$ represents the fraction of newborns who acquire the disease from the mothers. Finally, infected predators suffer an additional disease-related mortality $\mu$. Studying the model and the Jacobian matrix, we have found the conditions on the existence and stability of the equilibrium points (Table 1).

In the bifurcation analysis, we have studied the impact of pack hunting and of the disease. To this end, we fixed the other parameters and varied the cooperation $\alpha$ and transmissibility $\beta$. We summarize the results obtained in Fig 1 that shows a two-parameter bifurcation diagram. The figure shows different scenarios, in particular bistability between the equilibria $E_{ni}$ and $E_{npi}$ has been identified. This corresponds to a strong Allee effect induced by pack hunting.We can observe that it persists even in the presence of the disease. Whether or not predators survive depends on the initial conditions. The critical population density above which predators will survive is called the Allee threshold.

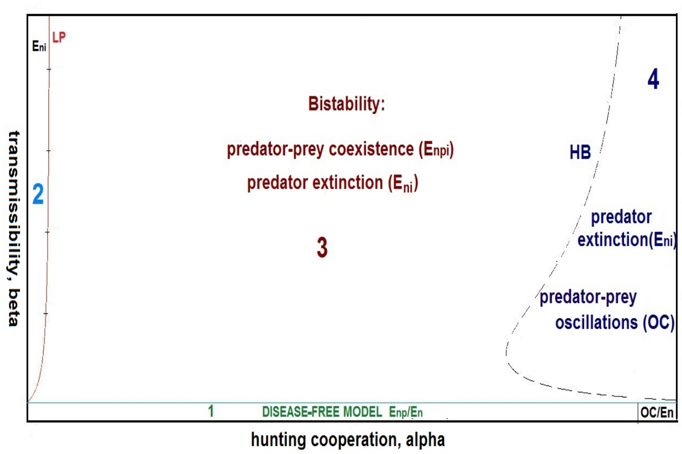


**FIGURE 1.** Two-parameter bifurcation diagram of model (1) in the parameter plane $(\alpha,\beta)$. The red line indicates limit point (LP) bifurcations and the dashed thin black line indicates Hopf bifurcations (HB). There are four different scenarios: (1) the disease-free system; (2) disease-induced extinction of predators; (3) crossing the red line bistability occurs and either the populations can coexist ($E_{npi}$) or the predators go extinct ($E_{ni}$); (4) bistability with either oscillatory coexistence (OC) or predators can still go extinct. The remaining parameters values are $k = 0.8$, $r = 10$, $\mu = 0.3$, $\theta = 0.1$.

However, from the analysis so far, it is not possible to understand how the Allee threshold changes. Therefore, we now introduce a method to study the separatrix surface.

## SEPARATRIX SURFACE

To approximate the separatrix it is necessary to detect the points lying on the curve. We start by considering a grid of $N^3$ initial condition points $P_j$ on the cubic domain $[0, l]^3$. The idea is to apply the bisection algorithm to the couple of points whose trajectories are different [2].
In our model the separatrix represents mathematically, the critical threshold for the predator population $p$. Therefore we expect that, in the space $npi$, the curve is almost parallel to the plane $n - i$.
Thus we apply the bisection method considering only the following initial conditions:

$$P_{1j} = (n_j, 0, i_j) \quad P_{2j} = (n_i, l, i_j) \quad j = 1, ..., N$$

Finally, we reconstruct the surface using a local Moving Least Squares (MLS) approximation. The MLS approximation represents a valid method as an alternative to the RBF interpolation. The basic idea, in fact, is to solve a locally weighted least squares problem for each evaluation point instead of considering a single large system and it does not require a big number of initial data [5]. Suppose that discrete values of a function $f$ are given at certain data sites $X = \{x_i, i = 1, ..N\} \in \mathbb{R}^s$ following the Backus-Gilbert approach [4] the approximation $u_f(x)$ to $f$ is represented by the quasi-interpolant:

$$u_f(x) = \sum_{i=1}^{N} f(x_i)\psi(x_i), \tag{2}$$

where $\psi(x_i)$ are the generating functions.
Let $U = span\{p_i, ..., p_m\}$, $m < N$, the approximation space with multivariate polynomials $p_m \in \prod_d^s$ of degree at most $d$. When $m, s < 3$ the Lagrange multipliers $\lambda_j$, $j = 1, ...m$ are founded explicitly, and the functions $\psi(x_i)$ can be expressed as:

$$\psi(x_i) = \omega(x_i) \sum_{j=1}^{m} \lambda_j p_j(x), \tag{3}$$

where $\omega$ represents the weighted function governing the influence of the data. In this paper we use the Wendland C2 compactly supported function: $\omega(r) = (1 - \epsilon r)_+^4 (4\epsilon r + 1)$, where $r$ is the distance $\|x - x_i\|$ and $\epsilon$ is the shape parameter. In Fig. 2 a summary of the algorithm is presented.

## NUMERICAL RESULTS AND CONCLUSIONS

In this section we present some results obtained by applying the MLS approximation of the separatrix in studying the Allee threshold. The model 1 is studied by fixing the biological parameters $r = 10$, $k = 0.8$, $m = 0.3$, $\theta = 0.1$ and the initial conditions on the domain $[0, 1]^3$, subdivided in $n = 5$ points on each edge, need for the proposed algorithm.
Then we evaluate the MLS approximant using the Wendland C2 function with the shape parameter $\epsilon = 5.5$.

```
Step 1    for j=1:N^{2}

Step 2     if P1j-->E_ni &  P2j-->E_npi
                then Point=bisection(P1(j),P2(j))

Step 3   Computation of the Lagrange multipliers
              and the generating functions

Step 4 Separatrix= MLS(Point);
```

**FIGURE 2.** Sketch of the algorithm to detect and reconstruct the separatrix surface

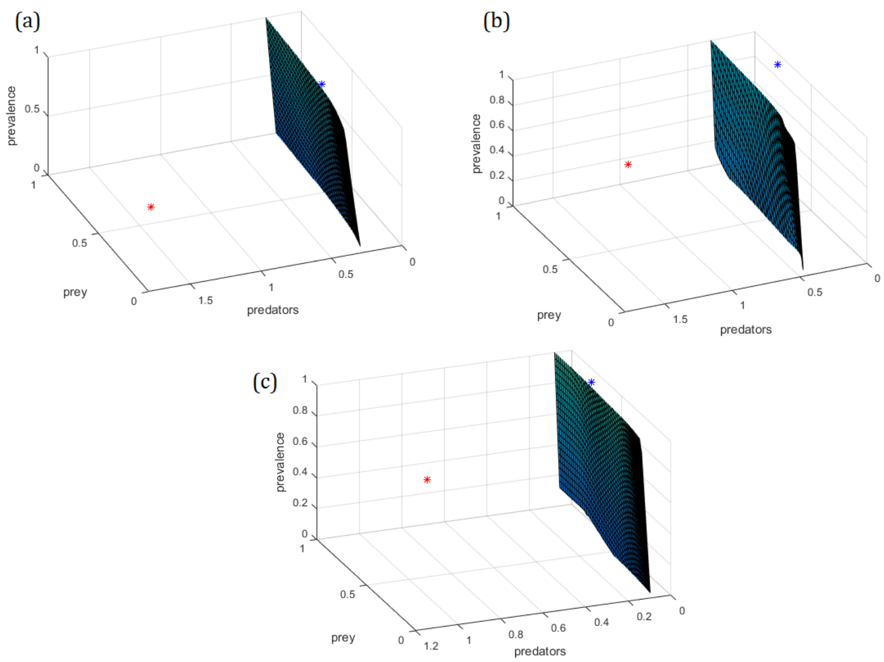


**FIGURE 3.** Approximation of the separatrix curve for different value of $\alpha$ and $\beta$. (a) $\alpha = 2, \beta = 2$, (b) $\alpha = 2, \beta = 8$ and (c) $\alpha = 6, \beta = 2$. The red point represents $E_{ni}$ and the blue one is $E_{npi}$.

Fig 3 shows three different cases obtained by varying the cooperation parameter $\alpha$ and the disease transmissibility $\beta$. We can observe that, when increasing $\alpha$ (Fig. 3(a)-(c)), the basin of attraction of the extinction point becomes smaller and the separatrix surface moves closer to the plane $p = 0$. This study supports the already established result that cooperation represents a fundamental behavior for the survival of the predators. In particular we can state that, for a wide range of values for $\alpha$, the populations almost always coexist unless the initial predator density is very small. Thus, even if pack hunting induces a strong Allee effect, for strong cooperation we do not find a significant critical predator density. In such cases, the Allee effect is more similar to a so-called weak Allee effect, in which there is no critical population density. Of course, if $\beta$ increases (Fig. 3(b)) we have the opposite situation. In fact, if the transmissibility becomes larger, the predators' chance to survive decreases because of the additional disease-induced mortality.

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