Physics Letters A 375 (2011) 3986–3992

Contents lists available at SciVerse ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Target-oriented chaos control

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ARTICLE INFO

Article history:
Received 4 April 2011
Received in revised form 2 August 2011
Accepted 3 August 2011
Available online 21 September 2011
Communicated by C.R. Doering

Keywords:
Chaos control
One-dimensional map
Linear feedback
Bifurcation
Population dynamics

ABSTRACT

Designing intervention methods to control chaotic behavior in dynamical systems remains a challenging problem, in particular for systems that are difficult to access or to measure. We propose a simple, intuitive technique that modifies the values of the state variables directly toward a certain target. The intervention takes into account the difference to the target value, and is a combination of traditional proportional feedback and constant feedback methods. It proves particularly useful when the target corresponds to the equilibrium of the uncontrolled system, and is available or can be estimated from expert knowledge (e.g. in biology and economy).

1. Introduction

Controlling chaos is an active field of research [1–5] that has led to the development of subtle control approaches which have been successfully applied in areas as diverse as physics [6], engineering [7], chemistry [8], medicine [9,10] and ecology [11–14]. Many systems, however, are difficult to control with existing methods in the sense that system parameters cannot be accessed or precisely measured, the system equations are unknown, or interventions are costly or only occasionally possible. This has prompted the need for even simpler and more robust control approaches [15,14,16,5]. Here, we propose such a method that is able to stabilize the system toward a certain target level over a wide range of values of the control parameter and, in addition, is more efficient in the long-run than other approaches.

The underlying idea of our method is rather simple: at each time-step, the control applies modifications directly to the state variable and takes into account the difference to the desired target level. This is a much more "target-oriented" approach than classical constant feedback (CF) [12,17–19], proportional feedback (PF) [20,21,5] or prediction-based control (PBC) [34,35] methods. Their control output is respectively constant or proportional to the state variable or takes into account the difference to the predicted dynamics. In particular, they do not take into account the desired target level of the control.

In the following, we devise our target-oriented approach on the basis of unimodal maps, which are classical prototypes of simple models being able to generate chaotic behavior [11,22]. They also have a clear biological interpretation as describing the dynamics of populations with non-overlapping generations [11]. Moreover, discrete-time maps arise naturally as return maps of higher-dimensional systems and can therefore be seen as simplified descriptions of more complicated systems [23]. For example, they have been shown to fit well complex time-series data originating from predator–prey systems or the spread of measles epidemics [24,25].

2. The method of target-oriented control (TOC)

Consider a map $N_{t+1} = f(N_t)$ that displays chaotic dynamics. Let N_t be the size of the state variable at time-step t. For simplicity, we will assume the variable is a population size, each time-step being a generation, although this same method could apply to any chaotic system, including for example magnetoelastic ribbons, diode resonators, chaotic lasers, particle accelerators, telecommunication, cardiac rhythms or economic markets [6,26–33].

Our aim is to introduce an intervention that induces stability over a large range of parameter values. We do this by using the intuitive approach of fixing a desired target value, T, and introducing individuals if the population size is below the target and removing individuals if it is above. The number we introduce or remove is proportional to the difference between the population and the target. Thus, the intervention involves

$$I(N) := c(T - N)$$

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doi:10.1016/j.physleta.2011.08.066

individuals, a value which we will refer to as the intervention. $c > 0$ is a proportionality parameter that we may choose, and will refer to simply as the *control*. Thus the greater the value we choose for the control c, the greater the intervention will be at each time-step. With this intervention at each time-step pre-reproduction our population will follow the model

$$N_{t+1} = f\big(N_t + I(N_t)\big) = f\big(cT + (1-c)N_t\big).$$

For certain parameter values this model can predict negative values for the population size, which we interpret as extinction. When this happens we therefore set the population size to be zero.

If the target corresponds to an unstable fixed point of the uncontrolled system, i.e. $T = N^*$ with $N^* = f(N^*)$ and $|f'(N^*)| > 1$, one can show that stabilization is possible if the control parameter is chosen from the range

$$1 - \frac{1}{|f'(N^*)|} < c < 1 + \frac{1}{|f'(N^*)|}. \tag{1}$$

More specifically, due to the assumption of chaotic dynamics in the absence of intervention, we can assume that N^* is oscillatorily unstable in the uncontrolled system, i.e. $f'(N^*) < -1$. Stabilization is then achieved via a period-halving (flip) bifurcation at $c = 1 - \frac{1}{|f'(N^*)|}$.

Condition (1) guarantees local stability, but there may be multiple attractors in the controlled system (see the following section) so that the target is not necessarily globally stable. Note that the parameter range (1) facilitating stabilization is an interval about 1 and that it always exists.

The target-oriented control scheme bears some similarities to the nonlinear feedback control method

$$N_{t+1} = f(N_t) - c\big(f(N_t) - N_t\big), \quad c \in [0,1]. \tag{2}$$

This is the "optimal control technique" proposed in [34, Section III]; for recent results and generalizations see [35]. We will refer to this method as *prediction-based control* (PBC) as any intervention requires knowledge of the future state of the system. By contrast, our approach aims to steer the system directly toward the desired state, which is why we refer to it as *target-oriented control* (TOC). Please note that this is different from the method of *targeting* [36].

In the next section, we focus on the case that the target is the unstable fixed point of the uncontrolled system, as this seems like the most natural choice. In the subsequent section, we investigate arbitrary target values. By way of example, we will apply our method to the exponential map (also known as the Ricker model):

$$N_{t+1} = f(N_t) = N_t e^{r(1-N_t)}, \tag{3}$$

where r is the reproduction parameter. This model exhibits a cascade of period doublings as r increases, until it demonstrates mostly chaotic dynamics when $r > 2.692$ [11]. We consider only the case when the uncontrolled system is chaotic as this is our topic of interest.

3. Targeting the original fixed point

In this section, the target of our intervention is the non-trivial fixed point, which has been scaled to one. Biologically speaking, it corresponds to the carrying capacity, which is the maximum population level naturally sustained by the environment. Hence with intervention, the dynamics are described by:

$$\begin{aligned} N_{t+1} &= f\big(N_t + I(N_t)\big) \\ &= \big(N_t + I(N_t)\big)e^{r(1-(N_t+I(N_t)))}, \end{aligned} \tag{4}$$

where $I(N_t) = c(1 - N_t)$.

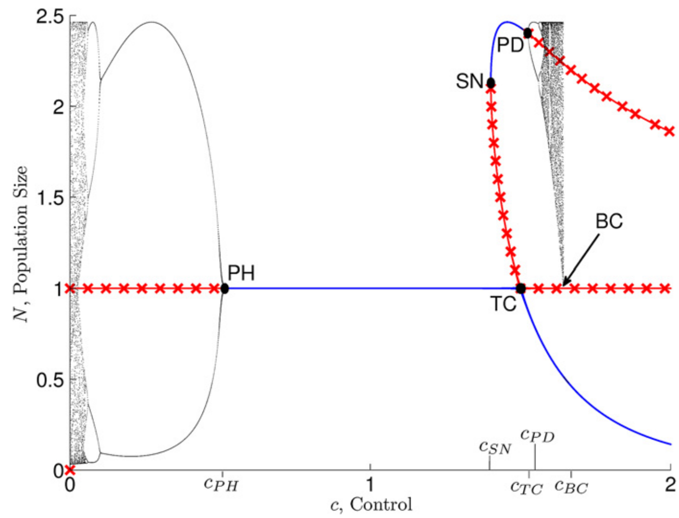

Fig. 1. (Color online.) Bifurcation diagram for model (4) with varying c. The bold blue lines indicate stable fixed points, the crossed red lines indicate unstable fixed points, and the black dots show periodic and chaotic attractors. The bifurcation points are indicated: PH = period halving, SN = saddle-node, TC = transcritical, PD = period doubling and BC = boundary crisis. Parameter value: $r = 3$.

The steady states of the model with intervention (4), their regions of stability and the bifurcation points are shown in Fig. 1. First, we can see that as c increases from $c = 0$ to c_{PH} we achieve stabilization via a cascade of period halvings to a fixed point, which remains constant at the carrying capacity 1. We shall call this fixed point N_1^*. This implies that once a population has reached the maximum level that the environment can support, it remains at this level.

N_1^* is stable for $c_{PH} = \frac{r-2}{r-1} < c < \frac{r}{r-1} =: c_{TC}$, cf. condition (1), which is a relatively large parameter range (Fig. 1). In practice, an intervention strategy will be subject to (natural) variations and noise in other parameters as well, such as the growth rate. A two-parameter bifurcation diagram reveals that the stability range is quite robust for realistic parameter values (see the electronic supplementary material). This is important because it may not be possible to control or measure them precisely in a natural environment.

Second, at a critical value of the control c_{SN}, there emerge two more fixed points at a saddle-node bifurcation. We label the lower and higher points N_-^* and N_+^*, respectively. Initially both are above the carrying capacity, however once c increases beyond c_{TC}, the lower fixed point drops below the carrying capacity and tends toward zero for large c. c_{TC} is the value of c corresponding to a transcritical bifurcation, where N_1^* and N_-^* exchange stability.

For values of the control between c_{SN} and c_{PD}, the higher fixed point N_+^* is stable and we have bistability. Whether the population size is attracted to the higher fixed point or the lower fixed point depends on the initial condition. This means that it is theoretically possible to maintain the population at N_+^*, a size greater than the carrying capacity of the system. Although this may appear counterintuitive, it can be explained by the ecological *hydra effect* [37–39, 5], in analogy to the mythological beast that grows two new heads for every one lost. Our intervention method renders this possible because at each generation we remove $c(N_t - 1)$ individuals from large populations. This reduces overcompensatory effects due to intra-specific competition between individuals. That is, individuals are free to flourish and the population can grow in number.

Finally c_{PD} is a period-doubling bifurcation point, after which N_+^* is no longer stable and we observe a cascade of period doublings toward chaos. However, at c_{BC} the chaotic attractor suddenly disappears. This happens when the attractor collides with the fixed

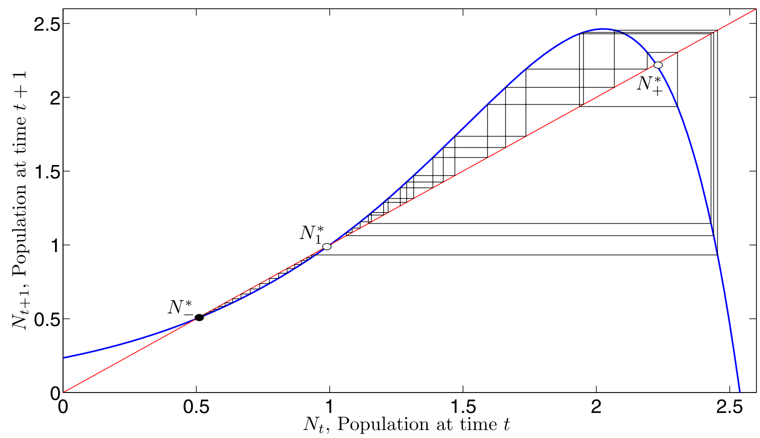

Fig. 2. (Color online.) Cobweb diagram of model (4) showing transient chaos. Iterations are initially in the basin of the chaotic attractor that existed before the boundary crisis but eventually stray below N^*_1 and are attracted down to N^*_-. Stable and unstable steady states are indicated with black and white circles respectively, for the case $r = 3$ and $c = 1.65$.

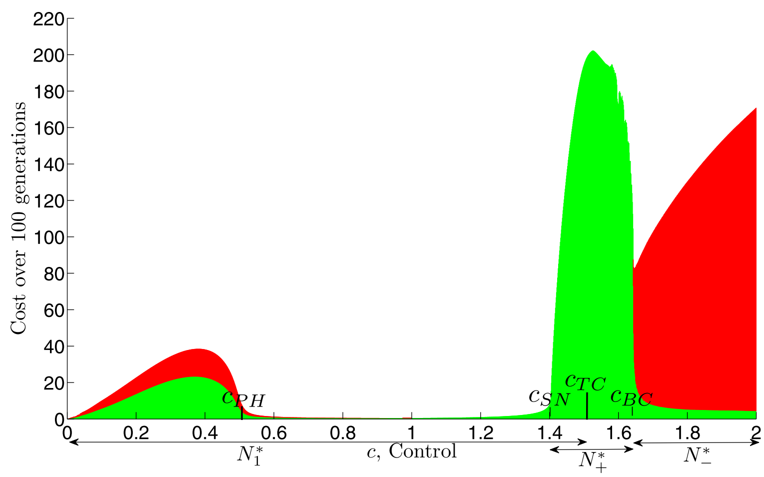

Fig. 3. (Color online.) Total cost incurred over 100 generations, where the cost at generation t is defined as $|I(N_t)| := |c(1 - N_t)|$, averaged across initial population values between 1 and 2. Green/light gray regions indicate the proportion of the intervention that was negative for those values of the control c, i.e. where the intervention at each generation involved the removal of individuals, and red/dark gray regions indicate the proportion of the intervention that was positive, so required the addition of individuals. The arrows indicate which stable fixed point that region of the graph is stabilizing to; recall initial values are between 1 and 2. The irregularities present on the upper boundary of the green/light grey region for c close to 1.6 are caused by transient chaotic behavior for early generations. When the cost is instead averaged across initial population values between 0 and 1, the graph obtained is the same, but without the large green/light gray hump seen here for the values of c where the model displays bistability. This is because the population size is then attracted down to N^*_-, requiring the addition of individuals. Here we have used $r = 3$.

point N^*_1 (cf. Fig. 1) and is known as a boundary crisis. Fig. 2 illustrates the cause of this behavior. As the value of c approaches c_{BC}, iterates of initial values close to N^*_+ fluctuate in widening ranges. Beyond c_{BC} the range is wide enough to include 1, so after some transient behavior iterations can stray into the basin of attraction for N^*_-. This causes the orbit to be attracted down to N^*_-, thus ending the chaotic dynamics.

Transient chaos prior to stabilization is undesirable in a practical sense, because it increases effort considerations and lengthens the time to control, which is our ultimate aim. We can compare cost and effort for different control strategies, i.e. different choices of the control c, by defining the cost per generation as the absolute value of the intervention, and summing these over a fixed number of generations. Fig. 3 shows the results over 100 generations for the target-oriented approach.

We can see that stabilization to the carrying capacity N^*_1 requires very little intervention in the long-term for $c_{PH} < c < c_{SN}$. This is because $I(N_t)$ approaches zero as the system approaches the target. Choosing $c = 1$ requires the least intervention in the long-term. This is somehow trivial because the intervention steers the system to the fixed point where it will remain in a deterministic system. Nevertheless, choosing $c \approx 1$ seems an optimal strategy for robust and efficient stabilization.

For comparison, Fig. 4 shows the effort required by three other control methods. First, consider PF where $N_{t+1} = f(cN_t)$ and $c \in \mathbb{R}$. Stabilization to a fixed point requires the control parameter to be reduced from $c = 1$ (no control) to at least $c = 0.4$. This incurs a cost of 160 (Fig. 4(a)). Second, consider CF where $N_{t+1} = f(N_t) + c$ and $c \in \mathbb{R}$. Stabilization requires the control parameter to be increased from $c = 0$ (no control) to at least $c = 0.9$, causing a cost of at least 100 (Fig. 4(b)). By contrast, with the target-oriented approach we have stabilization over a wide range of control values ($0.6 \leqslant c \leqslant 1.4$) for virtually diminishing costs (Fig. 3). This feature is shared by the PBC method (2), which is even more efficient for smaller values of c, but more costly close to $c \approx 1$ (Fig. 4(c)).

Theoretically, one may wish to stabilize the system to one of the alternative fixed points, i.e. N^*_+ or N^*_- rather than to the original fixed point N^*_1. Forcing the system to N^*_- may be both difficult and costly to implement since it requires augmentation at every generation. Much care would also be required to measure parameters and state variables with precision, to prevent extinction of the population which is rendered more probable since we are

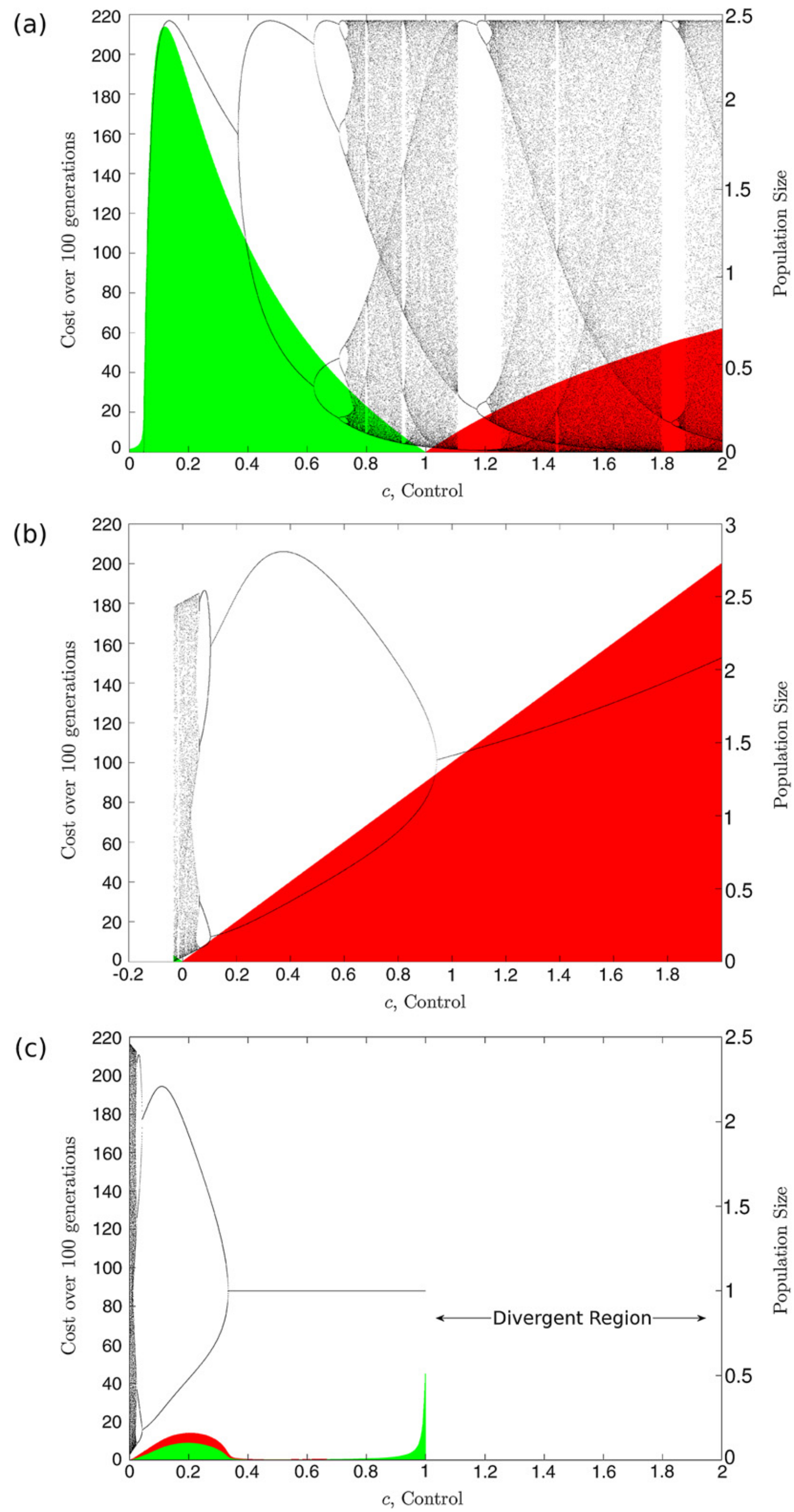

Fig. 4. (Color online.) Total costs and superimposed bifurcation diagrams for (a) the proportional feedback (PF) method, (b) the constant feedback (CF) method and (c) prediction-based control (PBC). The costs are incurred over 100 generations as in Fig. 3, with green/light gray regions indicating negative interventions (i.e. reduction of the state variable) and red/dark gray regions indicating positive interventions (i.e. augmentation of the state variable). The bifurcation diagrams in black show the orbital attractors. (a) In the PF method $c > 1$ corresponds to a proportional increase in the state variable and $c < 1$ to a proportional decrease in the state variable. (b) In the CF method $c > 0$ corresponds to positive constant feedback and $c < 0$ to a negative constant feedback. Note that for too large a negative constant feedback ($c \lesssim -0.05$), the state variable becomes negative, i.e. biological populations become extinct. This happens when there are more individuals removed than there actually are [18]. (c) PBC is defined for $c \in [0, 1]$. For $c > 1$, the system diverges, i.e. tends to infinity or takes negative values, depending on the initial condition. Parameter value: $r = 3$.

essentially maintaining the population size at an artificially low level. The accurate measurements required may not be possible should noise exist in the system.

Whilst stabilization to N_+^* requires a considerable amount of effort to reduce the state variable at each generation (the green/light gray hump in Fig. 3 between c_{SN} and c_{BC}), it may open up the way

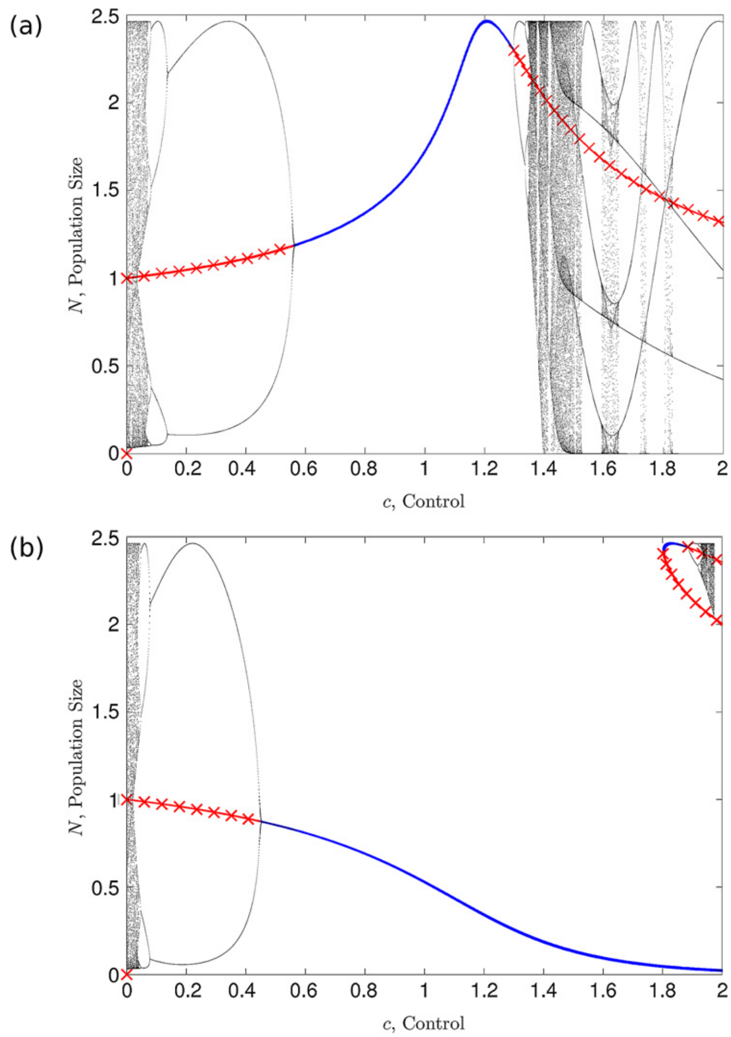

Fig. 5. (Color online.) Bifurcation diagrams for target values other than the carrying capacity. (a) Target smaller than the carrying capacity, $T = 0.7$, (b) target larger than the carrying capacity, $T = 1.3$. Bold blue lines represent stable fixed points and crossed red lines unstable fixed points. Black dots show asymptotic values arising from regular cycles or chaos. Parameter value: $r = 3$.

to new possibilities. In the population context, the removed individuals could be sold to third parties who may require the species, e.g. for restocking programs, medical research purposes or sterile insect release techniques for biological control of pest species. This means that the strategy to stabilize the population to N_+^* has the potential to be extremely profitable.

In the case $T = 1$, N_+^* is only stable for a relatively small range of values of the control c where we observe bistability in the system. This could be problematic should we wish to maintain the population size at the N_+^* value in a natural environment; slight perturbations or environmental noise could push the population level out of the basin of attraction of N_+^* and into that of N_1^* or N_-^*, depending on the value of c being used (see Fig. 1). The conditions on the noise variance for this to take place are derived in the electronic supplementary material.

4. Arbitrary target values

The analysis in the previous section is for the case where the targeted intervention is in relation to the carrying capacity ($T = 1$) of the system. Here we investigate the scenario when the target is chosen differently. That is, the intervention reads

$$I(N_t) = c(T - N_t) \tag{5}$$

with $T > 0$ and $T \neq 1$. We distinguish two cases, one where the target is smaller and one where the target is larger than the original fixed point. Fig. 5 shows the results for $T = 0.7$ and $T = 1.3$.

4.1. Target $T = 0.7$

We see a region of stability between ranges of c for which the single steady state is unstable. Depending on the control c chosen, the population size corresponding to the steady fixed point takes values anywhere between about 1.2 and just below 2.5. Thus it is theoretically possible to stabilize the population to any value in this range, whereas in the case where we target to the carrying capacity, a stable steady state at a population size greater than the carrying capacity only appears for a very small range of the control c.

The fact that, for the case where the target value is lower than the carrying capacity, the steady states appear at population values greater than the carrying capacity is again an example of the hydra effect. In other words, we observe greater population sizes when removing $c(N_t - 0.7)$ individuals, as this decreases the amount of intra-specific competition that would normally regulate population size.

Note that there is no bistability and consequently no boundary crisis for larger values of c. Instead, the system exhibits period doubling to chaos, followed by regions of periodic oscillations of low order.

4.2. Target $T = 1.3$

Conversely, targeting a value greater than the carrying capacity leads to stable fixed points corresponding to population sizes smaller than the carrying capacity. This is because here the intervention requires that we add $c(1.3 - N_t)$ individuals, more individuals than when targeting the carrying capacity. Thus at each generation we artificially increase the amount of intra-specific competition between the members of the population, and the numbers decrease. This is effectively a demonstration of behavior opposite to the hydra effect.

This intervention strategy, which enables stabilization to almost any population size less than the carrying capacity, has application where the suppression of a population is desired. For example, this may be relevant for pest species and invasive alien species. Fig. 5(b) suggests that such populations could be suppressed to any size smaller than the carrying capacity with this intervention method, assuming that without intervention the population is described by the Ricker map.

Similarly to the case $T = 1$, but unlike the case $T > 1$, a saddle-node bifurcation point appears and so there is a small range of control c for which we have bistability, followed by a period doubling bifurcation point and a boundary crisis.

5. Discussion and conclusions

The target-oriented control approach proves particularly useful when the target coincides with the fixed point of the uncontrolled system that we wish to stabilize. In fact, choosing the original fixed point as a target transpires naturally in many applications. For biological populations, the fixed point corresponds to the carrying capacity and can be considered as the desirable state. In chaotic lasers, as another example, the transition to chaos is usually from a known steady state [40].

Target-oriented control has three main advantages over proportional and constant feedback methods that lack a similar 'steering component'. First, stabilization to the desired state takes place not only for a single, particular control value (as in PF and CF control, see the bifurcation diagram in Fig. 4(a, b)) but for a wide range of values (see Fig. 1). Hence, the target-oriented approach to stabilization is robust over a considerably wider range of control values. Second, the effort required for stabilization is considerably less, which makes target-oriented control more efficient. Third, in practice it may not be clear how to choose the control parameter for the PF and CF method. There are theoretical results that disclose how to choose the control parameter [19,5], but they rely on knowing the underlying equations. Target-oriented control requires knowledge of only the fixed point, which may be gained from other sources or estimated by experts.

Target-oriented control has many common features with prediction-based control, most notably a strikingly similar pattern of efficiency (compare Figs. 3 and 4) and that stabilization to the fixed point takes place over a wide range of control parameters. PBC turns out to be advantageous for smaller values of the control parameter, as stabilization is attained for $c > (r - 2)/r$, while target-oriented control requires $c > (r - 2)/(r - 1)$. Moreover, interventions are more efficient. However, for larger values of the control parameter (around $c \approx 1$ where control is most robust), PBC is slightly more costly.

In comparison with prediction-based control, the advantage of TOC is that it only requires the target but does not need to be 'predictive' in the sense of knowing the underlying system dynamics. Liz and Franco [35] have pointed out that the model equation can be estimated when good census data of a population are available. However, such data would need to be based on observations over many generations, which is relatively rare in ecology except for some examples from fisheries or lab experiments [13]. Furthermore, it is inherently difficult to observe and therefore to reconstruct the stock–recruitment map at small population densities. By contrast, estimating the carrying capacity as a target for our control method appears comparably easy. Determining the carrying capacity from field data certainly requires less data, even if the population is cyclic [41], and there are a number of methods in place to estimate the carrying capacity utilizing other available data, e.g. [42,43]. Furthermore, many ecologists and resource managers have a good intuition about the carrying capacity, and this sort of 'expert knowledge' may be a valuable resource in determining the target value.

If the target does not coincide with the original fixed point, stabilization is still possible over a wide parameter range, but the system tends to depart from the desired value. Similarly to the PF and CF method [19,5], there exists a control value for which the target can be met—to find this control value may not be straightforward, however. Numerics suggest that for targets less than the carrying capacity, the system stabilizes to values greater than the carrying capacity. By contrast, for targets greater than the carrying capacity, the steady state takes values below the carrying capacity. It appears that these intervention strategies have the ability to stabilize the system to almost any desired value (from almost zero up to approximately two and a half times the carrying capacity in the case of Fig. 5). This could be used to achieve large (or small) stable values that were not possible in the uncontrolled system. The PF method has a similar feature (as was rigorously proven in [5]). This fact is not surprising because the TOC method with target $T = 0$ is precisely the PF method $N_{t+1} = f((1 - c)N_t)$. So, in some sense, the TOC method inherits this characteristic for small values of the target T.

The target-oriented approach is effectively a linear feedback method and can thus be seen as a combination of constant and proportional feedback methods. Note that, in contrast to standard CF and PF control, the intervention in our method can both be negative or positive, making our approach in general more efficient and versatile. This is illustrated in Fig. 3, where the system can be both diminished and augmented for a given control value. In contrast, PF and CF control methods are restricted to only one form of intervention (cf. Fig. 4). Obviously, target-oriented control programmes require the capability to reduce (e.g. by culling, trapping or biological and chemical agents) as well as to increase the population (e.g. by releasing stocked individuals, opening corridors from neighboring habitats or increasing refuges).

Similarly, the target-oriented method can be regarded as a generalization of simple limiter control [44,32,31]. Limiters apply a thresholding algorithm and are well known for their combination of stabilization and targeting, which significantly speeds up the process of forcing the trajectory to its desired aim. However, for certain limiters this method can be relatively costly in terms of intervention frequency and magnitude, and the stable fixed points that emerge are always smaller than the carrying capacity. The target-oriented approach uses not only thresholding from one side, but also augmentation toward the desired state from the other side. That is, we replace the 'limiter' by a more approachable target.

The initial cost of intervention, i.e. the control signal, may be large in target-oriented control. For many systems, however, occasional but substantial interventions are more realistic as well as practical. For example, populations in remote or difficult-to-access ecological systems can only be managed in occasional, expensive and labor-intensive field campaigns, or during certain seasons in the year [15,14,16,5]. In the long-term, however, the target-oriented approach to the fixed point requires only little effort and is highly efficient. This is because interventions are in relation to the target—rather than simply adding/removing a fixed amount or proportion, as for constant and proportional feedback methods respectively.

The results reported here are not confined to the exponential Ricker model; robust stabilization is also possible for other well-known maps. The corresponding bifurcation diagrams are strikingly similar (electronic supplementary material). This is a pleasing characteristic of this strategy and is in contrast for example to constant negative feedback (e.g. emigration models [18]), which simplify dynamics of the Ricker model for certain parameter values but not for the quadratic model or density-dependent emigration.

Recent experiments in the lab have demonstrated that interventions in biological populations [13,45], electrical systems, lasers and chemical reactions [27,28,8] can successfully control observed chaotic oscillations. Note that even in large-scale ecological field experiments, manipulations were able to reduce population fluctuations [46,47]. The intuitive nature of the target-oriented control proposed here and its effectiveness in controlling chaotic behavior suggest that it may prove useful in regulating irregular oscillations, particularly in disciplines such as ecology where interventions toward to an easily identifiable target are applicable. Similar prospects may exist, for instance, in fields like epidemiology, resource management or economy.

Acknowledgements

The authors would like to acknowledge B. Adams, G. Bell, J.J.C.H. Chua and anonymous referees for their comments and suggestions as well as support from the EU-ICT project Epiwork.

Appendix A. Supplementary material

See electronic supplementary material for (i) a two-parameter bifurcation diagram; (ii) the derivation of conditions on the noise variance for stochastic attractor switching to occur; and (iii) alternative population maps.

Supplementary material related to this Letter can be found online at doi:10.1016/j.physleta.2011.08.066.

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